64. The width ℓ of the pyramid measured at variable height z is seen to decrease from L at the base (where $z = 0$) to zero at the top (where $z = H$). This is a linear decrease, so we must have

$$\ell = L\left(1 - \frac{z}{H}\right) .$$

If we imagine the pyramid layered into a large number N of horizontal (square) slabs (each of thickness Δz) then the volume of each slab is $V' = \ell^2 \Delta z$ and the mass of each slab is $m' = \rho V' = \rho \ell^2 \Delta z$. If we make the continuum approximation ($N \to \infty$ while $\Delta z \to dz$) and substitute from above for ℓ, the mass element becomes

$$dm = \rho L^2 \left(1 - \frac{z}{H}\right)^2 dz .$$

We note, for later use, that the total mass M is given by $\rho L^2 H/3$ using the volume relation mentioned in the problem, but this can also be derived by integrating the above expression for dm.

(a) Using Eq. 9-9 we find

$$z_{\text{com}} = \frac{1}{M}\int z\,dm = \frac{3}{\rho L^2 H}\int_0^H z\rho L^2 \left(1 - \frac{z}{H}\right)^2 dz$$

where ρ and L^2 are constants (and, in fact, cancel) so we obtain

$$z_{\text{com}} = \frac{3}{H}\int_0^H \left(z - \frac{2z^2}{H} + \frac{z^3}{H}\right) dz = \frac{H}{4} = 36.8 \text{ m} .$$

(b) Although we could do the integral $\int dU = \int gz\,dm$ to find the work done against gravity, it is easier to use the conclusion drawn in the book that this should be equivalent to lifting a point mass M to height z_{com}.

$$W = \Delta U = Mgz_{\text{com}} = \left(\frac{\rho L^2 H}{3}\right) g\,\frac{H}{4} = 1.7 \times 10^{12} \text{ J} .$$